JACOBI'S CONDITION FOR THE PROBLEM OF LAGRANGE IN THE CALCULUS OF VARIATIONS*

BY

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Introduction

The problem of Lagrange, in the calculus of variations, is that of minimizing an integral

$$J = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx \tag{1}$$

with respect to curves which join two fixed points, 1 and 2, and satisfy a system of differential equations of the form

$$\phi_\beta(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (\beta = 1, 2, \cdots, m, m < n). \tag{2}$$

For this problem proofs of the necessity of the Euler-Lagrange Rule and the conditions of Weierstrass and Legendre are obtainable without the use of the second variation.† For the Jacobi condition, however, the situation is less satisfactory. Kneser's proof‡ of this condition, which is based on the theory of envelopes, excludes important special cases, while the proofs that utilize the second variation, though more inclusive,§ involve elaborate and complicated transformations. Similar remarks apply to the derivations of the Legendre condition by Clebsch|| and von Escherich,¶ both proofs being based on transformations of the second variation. Hahn's deduction** of the Legendre condition from that of Weierstrass is more direct.

The object of this paper is to secure inclusive proofs of the necessity of the Legendre and Jacobi conditions by means of the second variation, but without the use of complicated transformations. If the curves of the family

* Presented to the Society, April 21, 1916.

† Bolza, *Vorlesungen über Variationsrechnung*, pp. 558, 603.

‡ Bolza, loc. cit., p. 610.

§ In this connection see remarks by Bolza, loc. cit., p. 634.

|| Journal für Mathematik, vol. 55 (1858), p. 254.

¶ Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften zu Wien, vol. 107 (1898), p. 1267.

** Monatshefte für Mathematik und Physik, vol. 17 (1906), p. 295. See also Bolza, loc. cit., p. 607.

$$(3) \qquad y_i = y_i(x, e) \qquad (i = 1, 2, \cdots, n),$$

pass through the points 1 and 2, satisfy equations (2), and contain, for the parameter value $e = 0$, the minimizing curve E_{12},

$$y_i = y_i(x, 0),$$

then the function

$$(4) \qquad J(e) = \int_{x_1}^{x_2} f(x, y(x, e), y'(x, e))\, dx$$

must have a minimum for $e = 0$. The arc E_{12} must be an extremal, and there exists a set of functions $\lambda_\beta(x)$ ($\beta = 1, 2, \cdots, m$), such that if

$$F = f + \sum_\beta \lambda_\beta \phi_\beta,$$

then the derivatives $J'(O)$ and $J''(O)$ are expressible in the forms*

$$J'(O) = \int_{x_1}^{x_2} \sum_i (\eta_i F_{y_i} + \eta_i' F_{y_i'})\, dx,$$

$$J''(O) = \int_{x_1}^{x_2} \sum_{ik} (\eta_i \eta_k F_{y_i y_k} + 2\eta_i \eta_k' F_{y_i y_k'} + \eta_i' \eta_k' F_{y_i' y_k'})\, dx,$$

where the functions $\eta_i(x)$ are the variations of the family (3), defined by the relations

$$(5) \qquad \left(\frac{\partial y(x, e)}{\partial e}\right)_{e=0} = \eta_i(x).$$

If the arc E_{12} minimizes the integral (1) the first variation $J'(O)$ must vanish, and the condition

$$(6) \qquad J''(O) \geqq 0$$

must also be satisfied, at least in the normal case, for every set of functions $\eta_i(x)$ which satisfy the equations

$$(7) \qquad \Phi_\beta(x, \eta_1, \cdots, \eta_n, \eta_1', \cdots, \eta_n') = \sum_i \left(\eta_i \frac{\partial \phi_\beta}{\partial y_i} + \eta_i' \frac{\partial \phi_\beta}{\partial y_i'}\right) = 0,$$

$$(8) \qquad \eta_i(x_1) = \eta_i(x_2) = 0.$$

The condition (6) implies a problem of Lagrange in the $x\eta$-space of precisely the same type as the original problem in the xy-space. The integral to be minimized is $J''(O)$, and the equations of condition corresponding to equations (2) are the equations (7). To this $x\eta$-problem we apply the Euler-Lagrange Rule, the Weierstrass condition, and the corner-point condition,† all of which would necessarily be included in a complete discussion of the

* Bolza, loc. cit., p. 558 and p. 620.

† Bolza, loc. cit., p. 571.

Lagrange problem, and which, as was remarked above, may be derived without the use of the second variation. The application of the Weierstrass condition to the $x\eta$-problem just described furnishes a proof of the Legendre condition for the xy-problem which is even simpler than that of Hahn. The Euler-Lagrange Rule and the corner-point condition lead to a proof of the Jacobi condition which includes the exceptional cases of Kneser's method, and which is much more simple and direct than the proofs which are based on the usual elaborate transformations of the second variation.

1. Preliminary definitions and theorems

In this paper the following notations will be used:

$$\frac{\partial F}{\partial y_i} = F_i, \qquad \frac{\partial F}{\partial y_i'} = G_i, \qquad \frac{\partial \phi_\beta}{\partial y_i} = \phi_{\beta i}, \qquad \frac{\partial \phi_\beta}{\partial y_i'} = \psi_{\beta i},$$

$$\frac{\partial^2 F}{\partial y_i\, \partial y_k} = P_{ik}, \qquad \frac{\partial^2 F}{\partial y_i\, \partial y_k'} = Q_{ik}, \qquad \frac{\partial^2 F}{\partial y_i'\, \partial y_k'} = R_{ik}.$$

The various indices α, i, r, etc., will always have the ranges:

$$\alpha, \beta = 1, 2, \cdots, m; \qquad i, j, k = 1, 2, \cdots, n; \qquad r, s = 1, 2, \cdots, n - m;$$

$$h = 1, 2, \cdots, 2n; \qquad \nu = 1, 2, \cdots, 2n + m;$$

and the integral (1) and the equations (2) and (7) will be written in the abbreviated forms

$$J = \int_{x_1}^{x_2} f(x, y, y')\, dx; \qquad \phi_\beta (x, y, y') = 0; \qquad \Phi_\beta (x, \eta, \eta') = 0.$$

Concerning the functions f, ϕ_β, and the arc E_{12} the following hypotheses are made:

(A) *The arc E_{12} is of class* C'' throughout the interval (x_1, x_2), and passes through two fixed points* 1 *and* 2.

(B) *The functions $f(x, y, y')$, $\phi_\beta (x, y, y')$ are continuous and possess continuous partial derivatives of the first three orders in a region T of points (x, y, y'), containing the values (x, y, y') along E_{12} in its interior.*

(C) *The arc E_{12} minimizes the integral* (1) *with respect to curves which lie entirely within the region T, pass through the points* 1 *and* 2, *and satisfy the differential equations* (2).

(D) *The arc E_{12} is normal in every sub-interval (ξ_1, ξ_2) of (x_1, x_2), i. e., the only system of m functions $\lambda_\beta (x)$ of class C' in (ξ_1, ξ_2), which satisfies the equations*

* A curve is said to be of class $C^{(n)}$ if its coördinates, $y_i(x)$, are continuous and possess continuous derivatives of the first n orders.

$$\sum_\beta \left(\lambda_\beta \phi_{\beta i} - \frac{d}{dx} (\lambda_\beta \psi_{\beta i}) \right) = 0,$$

is $\lambda_\beta = 0$.

(*E*) *At every point of the arc* E_{12} *the condition*

$$|\psi_{\beta\alpha}| \neq 0$$

is satisfied.

An *admissible curve* is defined as a curve of class D',* which lies entirely within the region T, passes through the points 1 and 2, and satisfies equations (2). The *variations* of a one parameter family (3) are the functions $\eta_i(x)$ defined by equations (5). A set of functions $\eta_i(x)$ which satisfy equations (7) and (8) is called a set of *admissible variations,* and it may be proved† as a result of the hypotheses just made that the elements $\eta_i(x)$ of every solution of equations (7) and (8) are the variations of a one-parameter family (3).

Let the family (3) be a set of admissible curves containing the minimizing arc E_{12} for the parameter value $e = 0$. Substituting the functions $y_i(x, e)$ in the integral (1) we obtain the function $J(e)$ of equation (4). In terms of the notations described above the conditions on the derivatives $J'(O)$ and $J''(O)$ are expressible in the forms

$$(9) \qquad J'(O) = \int_{x_1}^{x_2} \sum_i (\eta_i F_i + \eta_i' G_i)\, dx = 0,$$

$$J''(O) = \int_{x_1}^{x_2} \sum_{ik} (\eta_i \eta_k P_{ik} + 2\eta_i \eta_k' Q_{ik} + \eta_i' \eta_k' R_{ik})\, dx \geqq 0,$$

and these equations must be satisfied for every set of admissible variations $\eta_i(x)$. A consequence of the vanishing of the first variation (9) is the well-known Euler-Lagrange Rule:

If E_{12} *is a minimizing arc there exists a set of multipliers* $\lambda_\beta(x)$ *such that at every point of* E_{12} *the equations*

$$(10) \qquad F_i - \frac{d}{dx} G_i = 0$$

are satisfied.

An arc of class C'', whose coördinates y_i with m functions $\lambda_\beta(x)$, of class C', satisfy equations (2) and (10), is called an *extremal.* Under the hypotheses made the following theorems may be proved:‡

THEOREM 1. *If* x' *and* x'' *are the abscissas of two arbitrary points* P' *and* P'' *on the arc* E_{12}, *then an* n *parameter family of curves*

* A curve of class $D^{(n)}$ is continuous and is composed of a finite number of arcs, each of which is of class $C^{(n)}$.

† Bolza, loc. cit., p. 568.

‡ Bolza, loc. cit., pp. 604, 565.

$$y_i = y_i(x, b_1, \cdots, b_n)$$

can always be found, with the following properties:

(*a*) *The family contains the arc* E_{12} *for the values* $(b_1, \cdots, b_n) = (0, \cdots, 0)$, $x_1 \leqq x \leqq x_2$, *as indicated by the conditions*

$$y_i(x, 0, \cdots, 0) = y_i(x).$$

(*b*) *The functions* y_i, y_i', $\partial y_i/\partial b_k$, $\partial^2 y_i/\partial x \partial b_k$ *are continuous within a certain neighborhood of the values* $(x, b_1, \cdots, b_n)$ *defining the arc* E_{12}.

(*c*) *The curves all pass through the point* P', i. e.,

$$y_i(x', b_1, \cdots, b_n) = y_i(x').$$

(*d*) *The functions* $y_i(x, b_1, \cdots, b_n)$ *satisfy equations* (2).

(*e*) *For the values* $x = x''$, $(b_1, \cdots, b_n) = (0, \cdots, 0)$, *defining the point* P'' *the condition*

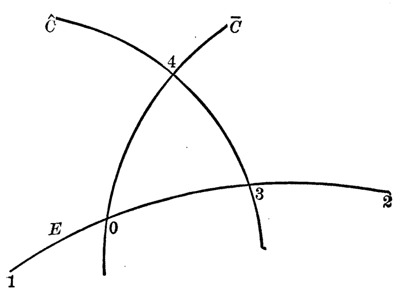
$$\left| \frac{\partial y_i}{\partial b_k} \right| \neq 0$$

is satisfied.

THEOREM 2. *To every normal extremal arc of class* C'' *there belongs an unique set of multipliers* $\lambda_\beta(x)$.

With the aid of Theorem 1 the Weierstrass condition* may now be derived. Through an arbitrary point 3 on the arc E_{12} a curve $\hat{C}$ is drawn,

$$y_i = \hat{y}_i(x),$$



whose coördinates are of class C' and satisfy equations (2). The values $\hat{y}_i'(x_3)$ may be assigned arbitrarily, except that the set of values x_3, $\hat{y}_i(x_3)$, $\hat{y}_i'(x_3)$ must lie entirely within the region T and satisfy condition (E). Let 4 be a point on the curve $\hat{C}$ whose abscissa is $x_4 = x_3 - e$, $(e > 0)$. By Theorem 1 we can draw through a point O on the arc E_{12} a curve $\bar{C}$ of class C',

$$y_i = \bar{y}_i(x, e),$$

which satisfies equations (2), passes through the point 4,

$$\bar{y}_i(x_4, e) = \hat{y}\quad x_4),$$

and reduces to the arc E_{12} for the value $e = 0$.

The one-parameter family V,

$$V = E_{10} + \bar{C}_{04} + \hat{C}_{43} + E_{32},$$

* Bolza, loc. cit., p. 603.

is a set of admissible curves. Since all the arcs satisfy equations (2), f may be replaced by F, and the function $J(e)$ for the family V expressed in the form

$$J(e) = \int_{x_1}^{x_0} F dx + \int_{x_0}^{x_4} \bar{F} dx + \int_{x_4}^{x_3} \hat{F} dx + \int_{x_3}^{x_2} F dx,$$

where the multipliers $\lambda_\beta(x)$ are those belonging to the arc E_{12}. Now $J(e)$ must be an increasing function for the value $e = 0$, and consequently the condition $J'(O) \geqq 0$ must be satisfied. On evaluating $J'(O)$ with the aid of partial integration and equations (10) the following condition, due to Weierstrass, is obtained:

If the arc E_{12} minimizes the integral (1) *the condition*

$$E(x, y(x), y'(x), p, \lambda) = F(x, y(x), p, \lambda) - F(x, y(x), y'(x), \lambda) - \sum_i (p_i - y_i') G_i(x, y(x), y'(x), \lambda) \geqq 0$$

must be satisfied at every point P of E_{12} for all values $p_i = \hat{y}_i'$ belonging to an admissible arc through P.

2. A NECESSARY CONDITION AT A CORNER POINT

A continuous curve $y_i = y_i(x)$ is said to have a *corner point* at $x = x_3$ if the condition

$$y_i'(x_3 - 0) \neq y_i'(x_3 + 0)$$

is satisfied for at least one value of i. To derive a necessary condition that a curve with a corner point shall minimize the integral (1) suppose given two normal extremal arcs of class C'':

$$E_{13} : y_i = y_i(x), \qquad E_{32} : y_i = \bar{y}_i(x),$$

which join the fixed points 1 and 3, and 3 and 2 respectively. Let the curve E_{12} composed of the arcs E_{13} and E_{32} minimize the integral (1) with respect to admissible curves.

By Theorem 1 an n-parameter family of curves can be found,

$$y_i = y_i(x, b_1, \cdots, b_n), \tag{11}$$

which are of class C', pass through the point 1, contain the arc E_{12} for the values $(b_1, \cdots, b_n) = (0, \cdots, 0)$, satisfy equations (2), and for which the condition

$$\left| \frac{\partial y_i}{\partial b_k} \right| \neq 0 \tag{12}$$

is satisfied for the values $x = x_3$, $(b_1, \cdots, b_n) = (0, \cdots, 0)$ defining the point 3.

The parameters b_k may be determined as functions of new parameters e_i so that the conditions

$$(13) \qquad y_i(x_3, b_1, \cdots, b_n) = e_i + y_i(x_3)$$

are satisfied. For, corresponding to the values $(e_1, \cdots, e_n) = (0, \cdots, 0)$ there is a solution, $b_i = 0$, of equations (13). Since the condition (12) is satisfied for the values $x = x_3$, $(b_1, \cdots, b_n) = (0, \cdots, 0)$ it is possible to solve equations (13) for the b_i and obtain n functions

$$(14) \qquad b_i = b_i(e_1, \cdots, e_n),$$

which are of class C' within a certain neighborhood $|e_k| < d$ and satisfy the conditions $b_i(0, \cdots, 0) = 0$. Substitution of the functions (14) in equations (11) gives an n-parameter family of curves

$$(15) \qquad y_i = y_i(x, e_1, \cdots, e_n),$$

which pass through the point 1,

$$(16) \qquad y_i(x_1, e_1, \cdots, e_n) = y_i(x_1),$$

contain the arc E_{12} for the values $(e_1, \cdots, e_n) = (0, \cdots, 0)$, and satisfy the conditions

$$(17) \qquad y_i(x_3, e_1, \cdots, e_n) = e_i + y_i(x_3).$$

In the same manner we construct a second family of curves

$$(18) \qquad y_i = \bar{y}_i(x, e_1, \cdots, e_n),$$

which pass through the point 2,

$$(19) \qquad \bar{y}_i(x_2, e_1, \cdots, e_n) = \bar{y}_i(x_2),$$

contain the arc E_{12} for the values $(e_1, \cdots, e_n) = (0, \cdots, 0)$, and satisfy the conditions

$$(20) \qquad \bar{y}_i(x_3, e_1, \cdots, e_n) = e_i + y_i(x_3).$$

The arcs (15) and (18) form an n parameter family of admissible curves. It is therefore permissible to replace f by F and consider the integral

$$J(e_1, \cdots, e_n) = \int_{x_1}^{x_3} F(x, y(x, e), y'(x, e), \lambda)\, dx + \int_{x_3}^{x_2} F(x, \bar{y}(x, e), \bar{y}'(x, e), \bar{\lambda})\, dx,$$

in which the functions $\lambda_\beta(x)$ and $\bar{\lambda}_\beta(x)$ are the multipliers belonging to E_{13} and E_{32} respectively. Since $J(0, \cdots, 0)$ must be a minimum the conditions

$$(21) \quad \frac{\partial J}{\partial e_k} = \int_{x_1}^{x_3} \sum_i (\eta_{ik} F_i + \eta'_{ik} G_i)\, dx + \int_{x_3}^{x_2} \sum_i (\bar{\eta}_{ik} F_i + \bar{\eta}'_{ik} G_i)\, dx = 0$$

must be satisfied, where

$$\eta_{ik} = \frac{\partial y_i}{\partial e_k}, \qquad \bar{\eta}_{ik} = \frac{\partial \bar{y}_i}{\partial e_k}, \qquad (e_1, \cdots, e_n) = (0, \cdots, 0).$$

Apply now the usual integration by parts and use equations (10). Equations (21) then become

$$(22) \qquad \sum_i \eta_{ik} G_i \Big]_{x_1}^{x_3} + \sum_i \bar{\eta}_{ik} G_i \Big]_{x_3}^{x_2} = 0.$$

But from equations (16) and (19)

$$\eta_{ik}(x_1) = \bar{\eta}_{ik}(x_2) = 0,$$

and from equations (17) and (20)

$$\eta_{ik}(x_3) = \bar{\eta}_{ik}(x_3) = \delta_{ik},$$

where δ_{ik} is zero for $i \neq k$ and unity for $i = k$. On substituting these values in equations (22) we obtain

$$G_i(x, y, y', \lambda)|_{x_3} = G_i(x, \bar{y}, \bar{y}', \bar{\lambda})|_{x_3}.$$

The following necessary condition at a corner point may therefore be stated: *If the minimizing curve E_{12} has a corner point at $x = x_3$ the conditions*

$$G_i(x_3 - 0) = G_i(x_3 + 0) \qquad (i = 1, 2, \cdots, n)$$

must be satisfied.

3. The Legendre condition

If we write

$$(23) \qquad \omega(x, \eta, \eta') = \sum_{ik} (\eta_i \eta_k P_{ik} + 2\eta_i \eta'_k Q_{ik} + \eta'_i \eta'_k R_{ik}),$$

then the condition

$$(24) \qquad J''(O) = \int_{x_1}^{x_2} \omega(x, \eta, \eta')\, dx \geqq 0$$

must be satisfied for every set of admissible variations $\eta_i(x)$. The $x\eta$-problem implied in this condition is of precisely the same type as the original problem in xy-space. In the first place it is clear from equations (23) and (7) that ω and Φ_β, as functions of x, η, η', satisfy the continuity conditions (B). Furthermore, since

$$\sum_i \left[\lambda_\beta \frac{\partial \Phi_\beta}{\partial \eta_i} - \frac{d}{dx}\left(\lambda_\beta \frac{\partial \Phi_\beta}{\partial \eta'_i} \right) \right] = \sum_i \left[\lambda_\beta \phi_{\beta i} - \frac{d}{dx}(\lambda_\beta \psi_{\beta i}) \right],$$

it follows from hypothesis (D) that every extremal of the $x\eta$-problem is normal throughout the interval (x_1, x_2). We may therefore apply to the $x\eta$-problem the theory previously developed.

By the Euler-Lagrange rule the coördinates η_i of a curve which minimizes the integral $J''(O)$ must satisfy, with m functions μ_β of class C', the equations

$$\Omega_{\eta_i} - \frac{d}{dx}\Omega_{\eta_i'} = 0, \tag{25}$$

where

$$\Omega = \omega + \sum_\beta \mu_\beta \Phi_\beta .$$

Equations (25), when written out, form with equations (7) the system

$$\begin{aligned} &\sum_k \left[\eta_k P_{ik} + \eta_k' Q_{ik} - \frac{d}{dx}(\eta_k Q_{ki} + \eta_k' R_{ik}) \right] \\ &\qquad\qquad + \sum_\beta \left[\mu_\beta \phi_{\beta i} - \frac{d}{dx}(\mu_\beta \psi_{\beta i}) \right] = 0, \\ &\sum_i [\phi_{\beta i}\eta_i + \psi_{\beta i}\eta_i'] = 0. \end{aligned} \tag{26}$$

Equations (26_1), which are linear and homogeneous in η_i, μ_β and their derivatives, are known in the theory of the second variation as the Jacobi equations. It will be understood hereafter that the elements $u_i(x)$, $\rho_\beta(x)$ of a solution $(u; \rho)$ of equations (26) are of class C'' and C' respectively.

If now the condition (24) is satisfied the arc in $x\eta$-space,

$$C: \qquad \eta_i \equiv 0, \qquad x_1 \leqq x \leqq x_2,$$

must be a minimizing extremal for the integral $J''(O)$, since its coördinates η_i, with the multipliers $\mu_\beta \equiv 0$, satisfy equations (26) and give $J''(O)$ its minimum value zero. The uniqueness of the set of multipliers $\mu_\beta \equiv 0$ is a consequence of hypothesis (D). Since the arc C minimizes the integral $J''(O)$ in the $x\eta$-space the Weierstrass condition for the $x\eta$-problem,

$$E(x, \eta, \eta', \zeta, \mu) \geqq 0, \tag{27}$$

must be satisfied at every point P of the arc C for all values $\zeta_i = \eta_i'$ belonging to an admissible arc through P. The condition (27) for the integral (24) reduces to the condition that

$$\sum_{ik} R_{ik}\zeta_i\zeta_k \geqq 0, \tag{28}$$

at every point of C, and equations (26_2) become

$$\sum_k \psi_{\beta k}\zeta_k = 0, \tag{29}$$

where the arguments of R_{ik} and $\psi_{\beta k}$ are the values $x, y_i(x), y_i'(x), \lambda_\beta(x)$

defining the arc E_{12}. We have therefore the following analogue of the Legendre condition, due to Clebsch:

If the arc E_{12} minimizes the integral (1) *the condition* (28) *must be satisfied at every point of E_{12}, for all values ζ_i which satisfy equations* (29).

4. The Jacobi condition

In deriving the Jacobi condition by the methods of this paper the following lemma is useful:

Lemma 1. *Let $(u;\rho)$ be a solution of equations* (26) *such that the functions u_i, u_i', ρ_β, all vanish for some value ξ, $x_1 \leqq \xi \leqq x_2$. If the determinant*

$$(30) \qquad R(x,y,y',\lambda) = \begin{vmatrix} R_{i1} \cdots R_{in}, & \psi_{1i} \cdots \psi_{mi} \\ \psi_{\beta 1} \cdots \psi_{\beta n}, & 0 \cdots 0 \end{vmatrix}$$

is different from zero along the arc E_{12}, then

$$u_i(x) \equiv 0, \qquad u_i'(x) \equiv 0, \qquad \rho_\beta(x) \equiv 0, \qquad x_1 \leqq x \leqq x_2.$$

Differentiate equations (26_2) with respect to x. If $R \neq 0$ the resulting equations may then be solved with (26_1) for η_i'' and μ_β'. A system is thus obtained of the form

$$(31) \quad \frac{d\eta_i}{dx} = \eta_i', \qquad \frac{d\eta_i'}{dx} = g_i(x,\eta,\eta',\mu), \qquad \frac{d\mu_\beta}{dx} = h_\beta(x,\eta,\eta',\mu)$$

with second members linear and homogeneous in the variables η_i, η_i', μ_β. Every solution of (26) is also a solution of (31). Now it is well known* from the theory of differential equations that there is one, and but one, solution of equations (31) which assumes for a given value of x a prescribed set of initial values. In particular, there is an unique solution which assumes the values $(0,0,0)$ for $x=\xi$. Since this solution is evidently

$$\eta_i \equiv 0, \qquad \eta_i' \equiv 0, \qquad \mu_\beta \equiv 0,$$

the lemma follows at once.

Assume now that a solution $(u;\rho)$ of equations (26) can be found in which the functions $u_i(x)$ vanish for two values ξ_1, ξ_2,

$$(32) \qquad u_i(\xi_1) = 0 = u_i(\xi_2), \qquad x \leqq \xi_1 < \xi_2 < x_2,$$

without all being identically zero throughout the interval (ξ_1, ξ_2). The curve C in the $x\eta$-space defined by the conditions

$$\begin{array}{lll} \eta_i \equiv 0, & \mu_\beta \equiv 0, & x_1 \leqq x \leqq \xi_1, \\ \eta_i = u_i(x), & \mu_\beta = \rho_\beta(x), & \xi_1 \leqq x \leqq \xi_2, \\ \eta_i \equiv 0, & \mu_\beta \equiv 0, & \xi_2 \leqq x \leqq x_2, \end{array}$$

* Bolza, loc. cit., p. 171.

is an admissible curve for the $x\eta$-problem, and has corner points possibly at ξ_1 and ξ_2. If the curve C minimizes the integral $J''(O)$ the corner point condition

$$\Omega_{\eta_i'}(\xi_2 - 0) = \Omega_{\eta_i'}(\xi_2 + 0) \tag{33}$$

must be satisfied. Furthermore

$$\Phi_\beta(x, \eta, \eta')\big|_{\xi_2} = 0. \tag{34}$$

Since the values $\eta_i(\xi_2)$ are all zero, while

$$\eta_i'(\xi_2 - 0) = u_i'(\xi_2), \qquad \eta_i'(\xi_2 + 0) = 0,$$

the conditions (33) and (34) may be written

$$\left.\begin{aligned} \sum_k 2u_k' R_{ik} + \sum_\beta \rho_\beta \psi_{\beta i}\big|_{\xi_2} &= 0, \\ \sum_k 2u_k' \psi_{\beta k} &= 0. \end{aligned}\right. \tag{35}$$

If now the determinant (30) is different from zero at $x = \xi_2$ then the only solution of equation (35) is

$$u_k'(\xi_2) = 0, \qquad \rho_\beta(\xi_2) = 0. \tag{36}$$

But if equations (32) and the cornerpoint conditions (36) are satisfied it follows from Lemma 1 that

$$u_i(x) \equiv 0, \qquad \rho_\beta(x) \equiv 0, \qquad x_1 \leqq x \leqq x_2,$$

which contradicts the hypothesis that the functions $u_i(x)$ do not all vanish identically. Therefore the corner point conditions (36) are not satisfied, and consequently the curve C cannot minimize the integral $J''(O)$.

Consider next the value of the integral $J''(O)$ taken over the curve C. Since equations (26_2) are satisfied at every point of C, ω may be replaced by Ω and the integral $J''(O)$ for the $x\eta$-curve C expressed in the form

$$J''(O) = \int_{\xi_1}^{\xi_2} \Omega(x, u, u', \rho)\, dx. \tag{37}$$

Now by Euler's theorem on homogeneous functions

$$2\Omega = \sum_k (u_k \Omega_{u_k} + u_k' \Omega_{u_k'}) + \sum_\beta \rho_\beta \Omega_{\rho_\beta},$$

where the last sum vanishes, since

$$\Omega_{\rho_\beta} = \Phi_\beta = 0$$

along C. The integral (37) then becomes

$$J''(O) = \frac{1}{2}\int_{\xi_1}^{\xi_2} \sum_k (u_k \Omega_{u_k} + u'_k \Omega_{u'_k})\, dx.$$

After integrating by parts, and using equations (25), it follows that

$$J''(O) = \tfrac{1}{2} \sum_k u_k \Omega_{u'_k} \Big]_{\xi_1}^{\xi_2} = 0,$$

on account of the conditions (32).

Since the curve C does not minimize the integral $J''(O)$, and yet gives it the value zero, it follows that there must exist an admissible arc in the $x\eta$-space for which $J''(O)$ is negative and consequently the condition (24) not satisfied. In this event the arc E_{12} does not minimize the integral (1). We may therefore state the following analogue of the Jacobi condition:

Consider an arc E_{12} *that satisfies the conditions* (A), . . ., (D) *of section* 1, *and along which the determinant* (30) *is different from zero. Let* ξ_1, ξ_2 *be any two points*

$$x_1 \leqq \xi_1 \leqq x_2, \qquad x_1 < \xi_2 < x_2,$$

where the values x_1, x_2 *define the end points of the arc* E_{12}. *If* E_{12} *minimizes the integral* (1) *there can exist no solution* $(u\,;\rho)$ *of equations* (26) *with elements* $u_i(x)$ *all vanishing at the end points of the interval* (ξ_1, ξ_2) *but not identically zero within it.*

Two points ξ_1, ξ_2 with which there is associated a solution $(u\,;\rho)$ of equations (26) in which the functions $u_i(x)$ vanish at ξ_1 and ξ_2 without all being identically zero in (ξ_1, ξ_2), are said to be *conjugate*. If, in particular $x_1 = \xi_1$ the Jacobi condition may be stated:

If the arc E_{12} *minimizes the integral* (1) *no point* ξ *conjugate to* x_1 *can lie within the interval* (x_1, x_2).

5. Relations between the conjugate point and solutions of the Euler equations

In the literature of the calculus of variations the conjugate points are usually defined as the zeros of a determinant, the elements of which are obtained by differentiating the general solution of equations (2) and (10) with respect to the constants of integration. This definition furnishes methods of great practical value for determining conjugate points, and it is therefore desirable to show that these criteria may be obtained from the definition of conjugate points given in the preceding section. For this purpose it is convenient to prove the following lemmas.

Lemma 2. *If* $(z_{ih}\,;r_{\beta h})$ *is a system of* $2n$ *linearly independent solutions of equations* (26) *every other solution* $(u\,;\rho)$ *of these equations is expressible in the form*

$$u_i = \sum_h c_h\, z_{ih}, \qquad \rho_\beta = \sum_h c_h\, r_{\beta h}.$$

In the first place, the solutions of the system (31) satisfy the Jacobi equations (26_1) and the conditions

$$(38) \qquad \sum_i (\phi_{\beta i}\eta_i + \psi_{\beta i}\eta_i') = C_\beta .$$

By choosing the initial values of a solution of equations (31) properly it is always possible to find a solution $(z\,;\,r)$ of the Jacobi equations (26_1) in which the functions $z_i(x)$ satisfy equations (38) for a prescribed set of values C_β. Consequently we can adjoin to the $2n$ solutions $(z_{ih}\,;\,r_{\beta h})$ of the lemma m other solutions $(z_{i,\,2n+\alpha};\ r_{\beta,\,2n+\alpha})$ of the Jacobi equations, in which the functions $z_{i,\,2n+\alpha}$ satisfy the conditions

$$(39) \qquad \sum_i (\phi_{\beta i} z_{i,\,2n+\alpha} + \psi_{\beta i} z_{i,\,2n+\alpha}') = \delta_{\beta\alpha} .$$

The $2n + m$ solutions of equations (31) thus obtained form a fundamental system. To prove this, write the product of the determinants $|\psi_{\beta\alpha}|$ and D_1 where

$$|\psi_{\beta\alpha}| = \begin{vmatrix} \psi_{\beta\alpha} & \psi_{\beta r} & \phi_{\beta i} & 0 \\ 0 & \delta_{sr} & 0 & 0 \\ 0 & 0 & \delta_{ji} & 0 \\ 0 & 0 & 0 & \delta_{\alpha\beta} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} z_{i\nu}' \\ z_{i\nu} \\ r_{\alpha\nu} \end{vmatrix}.$$

In these determinants each element is a matrix whose dimensions are indicated by the subscripts attached, or else by its position. The ranges of the subscripts are those given at the beginning of section 1. The symbol δ_{ij} stands as usual for unity or zero according as $i = j$ or $i \neq j$.

Using the rows of $|\psi_{\beta\alpha}|$ and the columns of D_1, and remembering that the elements of the first $2n$ columns of D_1 are solutions of equations (26), while the elements of the last m columns satisfy equations (39), we obtain the following product determinant of order $2n$:

$$D_2 = \begin{vmatrix} z_{rh}' \\ z_{ih} \\ r_{\alpha h} \end{vmatrix}.$$

Suppose now that D_2 vanishes for some value x_3, $x_1 \leqq x_3 \leqq x_2$. Then it is always possible to find $2n$ constants c_h such that

$$(40) \quad \sum_h c_h z_{m+r,\,h}'(x_3) = 0, \qquad \sum_h c_h z_{ih}(x_3) = 0, \qquad \sum_h c_h r_{\beta h}(x_3) = 0.$$

Let

$$v_i(x) = \sum_h c_h z_{ih}(x), \qquad \sigma_\beta(x) = \sum_h c_h r_{\beta h}(x).$$

The functions v_i are solutions of equations (26_2). After substituting and putting $x = x_3$ it follows from (40) that

$$\sum_\alpha \psi_{\beta\alpha} v'_\alpha |_{x_3} = 0,$$

and consequently, since $|\psi_{\beta\alpha}| \neq 0$, that all the values $v'_\alpha(x_3)$ are zero. Since the functions v_i, v'_i, σ_β, all vanish at x_3 it follows from Lemma 1 that v_i and σ_β are identically zero. But this contradicts the hypothesis that the $2n$ solutions $(z_{ih}\,;\, r_{\beta h})$ are linearly independent. Therefore the determinants D_2 and D_1 cannot vanish anywhere in the interval (x_1, x_2), and consequently the $2n + m$ solutions $(z_{i\nu}\,;\, r_{\beta\nu})$ of equations (31) form a fundamental system.

Let $(u\,;\,\rho)$ be any solution of equations (26). Then it is also a solution of equations (31) and is expressible in the form

$$u'_i = \sum_\nu c_\nu z'_{i\nu}, \qquad u_i = \sum_\nu c_\nu z_{i\nu}, \qquad \rho_\beta = \sum_\nu c_\nu r_{\beta\nu}. \tag{41}$$

Multiply equations (41_1) and (41_2) by $\psi_{\beta i}$ and $\phi_{\beta i}$ respectively and add the resulting $2n$ equations. Since the functions u_i and z_{ih} are solutions of equations (26), while the functions $z_{i,\,2n+\alpha}$ satisfy equations (39), it follows that the constants $c_{2n+\alpha}$ are all zero. Therefore

$$u_i = \sum_h c_h z_{ih}, \qquad \rho_\beta = \sum_h c_h r_{\beta h},$$

which proves the lemma.

LEMMA 3. *If* $(z_{ih}\,;\, r_{\beta h})$ *is a system of* $2n$ *solutions of equations* (26), *and if the determinant*

$$\Delta(x, x_1) = \begin{vmatrix} z_{ih}(x) \\ z_{ih}(x_1) \end{vmatrix}$$

does not vanish identically, then the zeros of $\Delta(x, x_1)$ *determine the conjugate points to* 1.

First, let 3 be a point conjugate to 1. Then there is associated with the point 3 a solution $(u\,;\,\rho)$ of equations (26) in which the functions $u_i(x)$ vanish at x_1 and x_3 without all being identically zero throughout the interval (x_1, x_3). Since $\Delta(x, x_1) \not\equiv 0$ the $2n$ solutions $(z_{ih}\,;\, r_{\beta h})$ are linearly independent, and consequently by Lemma 2 the functions $u_i(x)$ are expressible in the form

$$u_i(x) = \sum_h c_h z_{ih}.$$

If now the functions $u_i(x)$ all vanish at x_1 and x_3 the conditions

$$\sum_h c_h z_{ih}(x_1) = 0, \qquad \sum_h c_h z_{ih}(x_3) = 0, \tag{42}$$

must be satisfied. This requires that $\Delta(x_3, x_1) = 0$. Therefore the abscissa x_3 of every point 3 conjugate to 1 is a zero of $\Delta(x, x_1)$.

Next, let $\Delta(x_3, x_1) = 0$. Then it is always possible to determine $2n$ constants c_h such that equations (42) are satisfied. If we define

$$u_i = \sum_h c_h z_{ih}, \qquad \rho_\beta = \sum_h c_h r_{\beta h},$$

we have a solution of equations (26) in which the functions $u_i(x)$ vanish at x_1 and x_3 and yet are not all identically zero in (x_1, x_3) on account of the hypothesis that $\Delta(x, x_1) \not\equiv 0$. Therefore every zero x_3 of the determinant $\Delta(x, x_1)$ is the abscissa of a point 3 which is conjugate to 1.

LEMMA 4. *If* $(z_{in}; r_{\beta n})$ *is a system of* n *linearly independent solutions of equations* (26) *such that for all values of* i *and* k

$$z_{ik}(x_1) = 0, \tag{43}$$

then every other solution $(u; \rho)$ *in which the functions* $u_i(x)$ *all vanish for* $x = x_1$ *is expressible in the form*

$$u_i = \sum_k c_k z_{ik}, \qquad \rho_\beta = \sum_k c_k r_{\beta k}.$$

The proof is quite similar to that of Lemma 2. In the present instance we adjoin to the n solutions $(z_{ik}; r_{\beta k})$ of the lemma n solutions $(z_{i,\, n+k}; r_{\beta,\, n+k})$ of equations (26), with the single hypothesis that

$$|z_{i,\, n+k}(x_1)| \neq 0, \tag{44}$$

and in addition the m solutions $(z_{i,\, 2n+\alpha}; r_{\beta,\, 2n+\alpha})$ of equations (31) which were used in the proof of Lemma 2. The $2n + m$ solutions of equations (31) thus obtained form a fundamental system. To show this form the product determinant D_2 and assume that it vanishes for some value x_3, $x_1 \leqq x_3 \leqq x_2$. By precisely the same argument as that used in Lemma 2 it follows that

$$\sum_h c_h z_{ih}(x) \equiv 0, \qquad \sum_h c_h r_{\beta h}(x) \equiv 0. \tag{45}$$

On putting $x = x_1$ in equations (45) and using (43) and (44) it will be seen that all the n constants c_{n+k} are zero. Equations (45) then become

$$\sum_k c_k z_{ik}(x) \equiv 0, \qquad \sum_k c_k r_{\beta k}(x) \equiv 0,$$

which contradicts the hypothesis that the n solutions $(z_{ik}; r_{\beta k})$ are linearly independent. Therefore the determinants D_2 and D_1 cannot vanish anywhere in (x_1, x_2), and consequently the $2n + m$ solutions $(z_{i\nu}; r_{\beta\nu})$ of equations (31) form a fundamental system. It follows at once from Lemma 2 that every solution $(u; \rho)$ of equations (26) is expressible in the form

$$u_i = \sum_h c_h z_{ih}, \qquad \rho_\beta = \sum_h c_h r_{\beta h}. \tag{46}$$

If the functions $u_i(x)$ all vanish at x_1 then

$$\sum_h c_h z_{ih}(x_1) = 0,$$

and from the relations (43) and (44) it follows that the n constants c_{n+k} are all zero. Equations (46) then become

$$u_i = \sum_k c_k z_{ik}, \qquad \rho_\beta = \sum_k c_k r_{\beta k}.$$

LEMMA 5. *If* $(z_{ik}; r_{\beta k})$ *is a system of* n *solutions of equations* (26) *in which the functions* $z_{ik}(x)$ *all vanish at* x_1, *and if the determinant*

$$D(x) = |z_{ik}|$$

does not vanish identically, then the zeros of $D(x)$ *determine the conjugate points to* 1.

First, let 3 be a point conjugate to 1, and $(u; \rho)$ the associated solution. By Lemma 4

$$u_i = \sum_k c_k z_{ik}.$$

Since $u_i(x_3) = 0$ the conditions

$$\sum_k c_k z_{ik}(x_3) = 0$$

must be satisfied, and therefore $D(x_3) = 0$, that is, the abscissa x_3 of every point 3 conjugate to 1 is a zero of $D(x)$. Again, suppose that

$$D(x_3) = 0.$$

In this event n constants c_k can be determined so that

$$\sum_k c_k z_{ik}(x_3) = 0.$$

If we define

$$u_i = \sum_k c_k z_{ik}, \qquad \rho_\beta = \sum_k c_k r_{\beta k},$$

we have a solution of equations (26) in which the functions $u_i(x)$ vanish at x_1 and x_3 and yet are not all identically zero, on account of the hypothesis $D(x) \not\equiv 0$. Therefore every zero x_3 of $D(x)$ is the abscissa of a point 3 which is conjugate to 1.

As consequences of the foregoing lemmas we have the customary criteria for determining the conjugate points, as follows:

THEOREM. *Let*

$$y_i = y_i(x, c_1, \cdots, c_{2n}), \qquad \lambda_\beta = \lambda_\beta(x, c_1, \cdots, c_{2n}) \tag{47}$$

be a $2n$ *parameter family of solutions of the Euler equations* (10) *and the equations of condition* (2), *containing the minimizing arc* E_{12} *and its multipliers for the values* $(c_1, \cdots, c_{2n}) = (0, \cdots, 0)$. *If the determinant*

$$\Delta(x, x_1) = \begin{vmatrix} \dfrac{\partial y_i(x)}{\partial c_h} \\[2ex] \dfrac{\partial y_i(x_1)}{\partial c_h} \end{vmatrix}$$

does not vanish identically along E_{12}, i. e., *when the constants* c_h *are all zero, then the zeros of* $\Delta(x, x_1)$ *determine the conjugate points to* 1 *on* E_{12}.

If the functions (47) are substituted in equations (2) and (10) identities in $x, c_1, \cdots, c_{2n}$ are obtained.* Differentiating with respect to any parameter c_h and then setting $(c_1, \cdots, c_{2n}) = (0, \cdots, 0)$ we obtain the well-known result that each of the $2n$ systems of functions

$$\frac{\partial y_i}{\partial c_h}, \quad \frac{\partial \lambda_\beta}{\partial c_h} \qquad (h = 1, 2, \cdots, 2n),$$

in which the constants c_h are all zero, is a solution of equations (26) formed for the extremal arc E_{12}. Since the linear independence of the $2n$ solutions is a consequence of the hypothesis that $\Delta(x, x_1) \not\equiv 0$ the theorem follows at once from Lemma 3.

THEOREM. *Let*

$$y_i = y_i(x, c_1, \cdots, c_n), \qquad \lambda_\beta = \lambda_\beta(x, c_1, \cdots, c_n)$$

be an n *parameter family of solutions of equations* (2) *and* (10), *containing the minimizing arc* E_{12} *for the values* $(c_1, \cdots, c_n) = (0, \cdots, 0)$, *and satisfying the conditions*

$$y_i(x_1, c_1, \cdots, c_n) = y_i(x_1). \tag{48}$$

If the determinant

$$D(x) = \left| \frac{\partial y_i(x)}{\partial c_k} \right|$$

does not vanish identically along E_{12} *then the zeros of* $D(x)$ *determine the conjugate points to* 1 *on* E_{12}.

The n systems of functions

$$\frac{\partial y_i}{\partial c_k}, \quad \frac{\partial \lambda_\beta}{\partial c_k} \qquad (k = 1, 2, \cdots, n),$$

where $(c_1, \cdots, c_n) = (0, \cdots, 0)$, are solutions of equations (26). The linear independence of these solutions follows from the hypothesis that $D(x) \not\equiv 0$. Furthermore, it is clear from equations (48) that

$$\frac{\partial y_i(x_1)}{\partial c_k} = 0.$$

The theorem is an immediate consequence of Lemma 5.

* In this connection see Bolza, loc. cit., pp. 72, 623.